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IMPERFECTION-SENSITIVITY OF CYLINDRICAL PANELS UNDER COMPRESSION USING KOITER'S IMPROVED POSTBUCKLING THEORY

DAVID HUI and Y. H. CHEN
Department of Engineering Mechanics, Ohio State University, Columbus, OH 43210, U.S.A.

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Abstract—This paper deals with the use of Koiter's improved postbuckling theory in axial buckling of integrally stiffened cylindrical panels. According to Koiter's improved theory, the postbuckling coefficients are evaluated at the actual applied load rather than at the classical buckling load. Substantial positive shift of the postbuckling is found which indicates that the imperfection-sensitivity predicted by Koiter's 1945 general theory may significantly overestimate the degrading effects of these imperfections. Such a positive shift is especially crucial in studying mode interactions such as local and overall buckling mode interactions of stringer-reinforced cylindrical shells.

1. INTRODUCTION

It is now well established that the major reason for the discrepancies between theoretical and experimental static buckling loads of thin shells is the presence of unavoidable geometric imperfections. Koiter (1945) was the first to present a rigorous postbuckling theory which is capable of taking into account the presence of geometric imperfections and interactions between buckling modes. Koiter's theory was not well known until his thesis was translated from Dutch into English in 1967, paving the way for a breakthrough in the theory of elastic stability. Numerous papers were written based on Koiter's theory as evidenced by a number of comprehensive reviews (Hutchinson and Koiter, 1970; Budiansky and Hutchinson, 1979; Citerley, 1982; Leissa, 1985; Simitses, 1986), books (Thompson and Hunt, 1973; Yamaki, 1984; Kollar and Dulacska, 1984; Bushnell, 1985) and IUTAM conference proceedings (Budiansky, 1974; Koiter and Mikhailov, 1980; Thompson and Hunt, 1983).

Koiter's 1945 general theory is based on an asymptotic perturbation technique and has several limitations. First, the general theory does not take into account the deformations prior to buckling and this limitation was later removed by Fitch (1968). Second, it was assumed that the shape of the imperfection is identical to that of the buckling mode and this was addressed by Tvergaard and Needleman (1983). Third, the theory is valid asymptotically only for sufficiently small values of the imperfection amplitude. Various authors have extended Koiter's theory to include random imperfections, plastic deformations and general computer codes were readily available (Bauld and Satyamurthy, 1979; Bushnell, 1985). However, the crux of the problem lies in the range of validity of the imperfection-sensitivity curves. Based on independent upper-bound calculations of the buckling loads of pressurized spherical shells (Hutchinson, 1967) and cylindrical shells (Koiter, 1963; Hutchinson, 1965), it was estimated that Koiter's 1945 general theory predicts reasonably good results for imperfection amplitudes up to one shell thickness. The abovementioned range of validity may be quite misleading since Koiter and Pignataro (1976) pointed out that the general theory is valid for imperfection amplitudes up to only a few percent of the skin thickness in the case of simultaneous buckling mode interaction of integrally stiffened flat plates under compression.

This paper was motivated by Koiter's remarks that the general theory of elastic stability may be improved. Koiter (1976) was the first to point out that, "A better accuracy, however, may be achieved for larger values of $|\lambda_1 - 1|$ and $|\lambda - 1|$, if each of the coefficients C_1 and C_2 is evaluated at the actual load factor λ, although we are unable to estimate the extended range of validity; the improvement (for the local mode) seems to be the most significant one because there is a strong dependence of C_2 on λ for all values of the flatness parameter which are not quite small; we recommend to evaluate both C_1 and C_2 at the actual values of the load factor in a systematic numerical evaluation of the theory". In the above Koiter's notation, λ_1 is the ratio of the overall buckling load divided by the local buckling load, C_1

is the transformed coefficient of the quartic term of the overall mode and C_2 is related to the quartic term of the local mode of an axially stiffened cylindrical shell under compression. It is particularly significant to note that among the large number of publications on the imperfection-sensitivity of structures (Khot, 1968; Imbert, 1971; Seide, 1974; Arbocz, 1974; Citerley, 1982, 1984, 1986; Yamaki, 1967, 1984), only one paper (Koiter, 1969) involves the "quantitative calculations" of the postbuckling coefficients at the actual load for shells. This is probably due to two factors: first, the researchers were unaware of such a possible improvement of Koiter's theory and second, the required computational effort to evaluate the coefficients at the actual applied load may be enormous. This paper is the first in the open literature to employ Koiter's "improved" stability theory to evaluate the postbuckling coefficients at the actual loads for an integrally stiffened cylindrical panel under compression. Particular emphasis is placed on an open cylindrical panel because this is part of a widely used stringer-reinforced closed cylindrical shell. Further, it was shown that accurate computations of postbuckling coefficients of these panels are essential for studying local and overall buckling mode interactions of axially stiffened cylindrical shells under compression (Tennyson *et al.*, 1980; Hui *et al.*, 1981; Hui, 1983, 1985b).

Evaluations of the postbuckling coefficients at the actual applied load rather than at the classical buckling load were first performed by Koiter (1969) who examined the non-linear buckling behavior of externally pressurized complete spherical shells. However, the interaction between many approximately simultaneous buckling modes makes it impossible to ascertain whether the imperfection sensitivity (or lack of it) is due to mode interaction or to the use of the "improved" theory. Nevertheless, it should be mentioned that the present work is based on the elegant technique employed by Koiter in his 1969 paper.

The analysis is based on a solution of Donnell-type non-linear equilibrium and compatibility of a cylindrical shell. Using Koiter's 1945 general theory, the non-linear differential equations can be written in terms of a sequence of linear differential equations for the buckling state and the second-order fields using an asymptotic perturbation technique. The boundary conditions for the integrally stiffened panel are obtained from the jump conditions arising from the natural boundary conditions using energy methods. Likewise, a sequence of linear boundary conditions is generated that is applicable to buckling and second-order problems. The analysis is simplified by assuming that the cylindrical panels are infinitely long, thus, permitting separation of variables and reduction of the partial differential equations to ordinary ones. These sets of ordinary differential equations and the boundary conditions are discretized using a central finite-difference scheme. The buckling loads and the associated buckling modes are obtained using the shifted inverse power method (Hui and Hansen, 1982). The second-order field non-homogeneous algebraic equations are solved using backward substitution. The postbuckling coefficients are computed based on a re-formulation of Koiter's 1945 theory by Budiansky and Hutchinson (1964). Special care is taken to ensure that (i) the differential equations for the second-order problem are solved by retaining the actual value of the applied load rather than evaluating the applied load at the classical buckling load; (ii) the postbuckling coefficients formulated by Budiansky and Hutchinson are also evaluated at the actual applied load. This procedure will yield the "improved" postbuckling coefficient as a function of the applied load.

A simple quadratic curve-fit of the improved imperfection-sensitivity curves is made for the imperfection amplitude between 0 and 0.25 times the shell thickness, enabling one to compute the improved postbuckling coefficient. In the event that the structure has stable postbuckling behavior (that is, insensitive to imperfection), the curve-fit scheme is applied to the initial stable postbuckling path for the amplitude of the buckling mode between 0 and 0.25 times the thickness. These improved coefficients are plotted as a function of the flatness parameter (Koiter, 1956) and stringer parameters such as torsional rigidity, axial stiffness, and eccentricity.

2. GOVERNING DIFFERENTIAL EQUATIONS AND CLASSICAL BUCKLING LOAD

The Donnell governing non-linear equilibrium and compatibility equations for a cylindrical shell are (see for example, Hutchinson (1965))

$$[Eh^3/(4c^2)](W_{,XXXX}+W_{,YYYY}+2W_{,XXYY})+(1/R)F_{,XX} = F_{,YY}W_{,XX}+F_{,XX}W_{,YY}-2F_{,XY}W_{,XY} \quad (1)$$

$$[1/(Eh)](F_{,XXXX}+F_{,YYYY}+2F_{,XXYY})-(1/R)W_{,XX} = (W_{,XY})^2-W_{,XX}W_{,YY}. \quad (2)$$

In the above equations, W is the out-of-plane deflection (positive outwards), F is the stress function, X and Y are the in-plane axial and circumferential coordinates, R is the shell radius, h is the skin thickness, E is Young's modulus of the skin and $c = [3(1-\nu^2)]^{1/2}$ where ν is Poisson's ratio. Introducing the following non-dimensional quantities ($q_0 = (2cR/h)^{1/2}$)

$$w = W/h, \qquad f = 2cF/(Eh^3), \qquad (x,y) = (q_0/R)(X,Y) \quad (3)$$

the governing non-linear partial differential equations become

$$w_{,xxxx}+w_{,yyyy}+2w_{,xxyy}+f_{,xx} = (2c)[f_{,yy}w_{,xx}+f_{,xx}w_{,yy}-2f_{,xy}w_{,xy}] \quad (4)$$

$$f_{,xxxx}+f_{,yyyy}+2f_{,xxyy}-w_{,xx} = (2c)[(w_{,xy})^2-w_{,xx}w_{,yy}]. \quad (5)$$

Koiter's general theory of elastic stability (Koiter, 1945) was re-formulated in terms of the mixed formulation involving the out-of-plane deflection and the stress function by Budiansky and Hutchinson (1964). Koiter's theory assumes that the total deflection and the total stress function can be expressed in the asymptotic form

$$(w,f) = (w_p,f_p)+(\xi w_I,\xi f_I)+(\xi^2 w_{II},\xi^2 f_{II}) \quad (6)$$

where ξ is the amplitude of the buckling mode normalized with respect to the skin thickness, subscript "p" refers to the pre-buckling, "I" refers to the buckling state and "II" stands for the second-order fields. Substituting the total deflection and stress function into the non-linear partial differential equations and collecting the linear terms in ξ, one obtains the linearized equilibrium and compatibility equations for the buckling state

$$w_{I,xxxx}+w_{I,yyyy}+2w_{I,xxyy}+f_{I,xx}+(2\sigma)w_{I,xx} = 0 \quad (7)$$

$$f_{I,xxxx}+f_{I,yyyy}+2f_{I,xxyy}-w_{I,xx} = 0 \quad (8)$$

where σ is the non-dimensional applied load (positive for compression). The dimensional applied stress $\bar{\sigma}$ (force per unit area) on the skin is the same as that on the stringers and σ is related to $\bar{\sigma}$ by

$$\sigma = \bar{\sigma}cR/(Eh) = -cf_{p,yy}. \quad (9)$$

The present analysis of axially compressed stringer-reinforced cylindrical panels will be simplified by assuming that the panels are sufficiently long such that the boundary conditions at the two curves edges may be neglected. Thus, the buckling mode can be written in the separable form

$$[w_I(x,y), f_I(x,y)] = [w_I(y), f_I(y)]\cos(Mx). \quad (10)$$

Substituting the separable form of the buckling mode into the linearized equilibrium and compatibility equations for the buckling state, one obtains

$$w_I(y)_{,yyyy}-2M^2w_I(y)_{,yy}+(M^4-2M^2\sigma)w_I(y)-M^2f_I(y) = 0 \quad (11)$$

$$f_I(y)_{,yyyy}-2M^2f_I(y)_{,yy}+M^4f_I(y)+M^2w_I(y) = 0. \quad (12)$$

This paper deals with the local buckling and postbuckling behavior of integrally stiffened closed cylindrical shells where the deformations occur primarily in the skin between adjacent stringers. At the stringer edge, the four jump conditions with respect to the axial, circumferential and the out-of-plane displacements U, V, W and the slope $W_{,Y}$ are respectively (Tvergaard, 1973; Hui, 1983)

$$(\delta U)\{N_{xy}^{+}-N_{xy}+(E_sA_s)[\varepsilon_x^{+}-e_sW_{,XX}^{+}+(1/2)(e_s)^2(W_{,XY}^{+})^2]_{,X}\}=0 \tag{13}$$

$$(\delta V)\{N_y^{+}-N_y^{-}-(E_sI_t)(V_{,XXXX}^{+})\}=0 \tag{14}$$

$$(\delta W)\{(-D)(W_{,YYY}^{+}-W_{,YYY}^{-})+e_sN_{s,XX}^{+}-E_sI_sW_{,XXXX}^{+}+N_s^{+}W_{,XX}^{+}\}=0 \tag{15}$$

$$(\delta W_{,Y})\{(D)(W_{,YY}^{+}-W_{,YY}^{-})+(G_sJ_s)(W_{,XXY}^{+})+(e_s)^2N_s^{+}W_{,XXY}^{+}+(e_s)^2N_{s,X}^{+}W_{,XY}^{+}\}=0. \tag{16}$$

The above jump conditions are presented in dimensional form for clarity purposes; the membrane stress resultants are related to the stress function by

$$N_x=F_{,YY},\qquad N_y=F_{,XX},\qquad N_{xy}=-F_{,XY}. \tag{17}$$

Further, E_s is Young's modulus of the stringer, A_s is the cross-sectional area of any one stringer, e_s is the stringer eccentricity measured from the skin middle surface to the centroid of the stringer, I_s is the out-of-plane moment of inertia of a stringer with respect to a circumferential line passing through the stringer centroid, I_t is the in-plane tangential moment of inertia of a stringer with respect to a radial line passing through the stringer centroid, D is the flexural rigidity, and G_sJ_s is the torsional rigidity of the stringer (Hui, 1983). Further, superscript "+" denotes the side of the stringer in the positive circumferential y-direction, while "−" denotes the other side of the stringer, and y is measured from any one stringer. The axial force applied at the stringer centroid is defined to be

$$N_s^{+}=(E_sA_s)(\varepsilon_s) \tag{18a}$$

where the longitudinal strain at the stringer centroid ε_s is related to the longitudinal strain at the skin middle surface stringer position ε_x^{+} by

$$\varepsilon_s=\varepsilon_x^{+}-e_sW_{,XX}^{+}+(1/2)(e_s)^2(W_{,XY}^{+})^2 \tag{18b}$$

where

$$\varepsilon_x^{+}=[1/(Eh)](F_{,YY}^{+}-\nu F_{,XX}^{+}). \tag{19}$$

Assuming a membrane pre-buckling state and that the stringer does not bend out of plane in the buckling state ($W_{I,XX}^{+}=0$), the axial force applied at the stringer centroid can be expressed as

$$N_s^{+}=N_s^{p}+\xi N_s^{I}+\xi^2N_s^{II}. \tag{20}$$

These assumptions are acceptable since the cylindrical panels being considered are infinitely long. Further, N_s^{p}, N_s^{I} and N_s^{II} are defined to be

$$\begin{aligned}N_s^{p}&=-E_sA_s\bar{\sigma}/E\\ N_s^{I}&=(E_sA_s)[(F_{I,YY}^{+}-\nu F_{I,XX}^{+})/(Eh)]\\ N_s^{II}&=(E_sA_s)\{[(F_{II,YY}^{+}-\nu F_{II,XX}^{+})/(Eh)]-e_sW_{II,XX}^{+}+(1/2)(e_s)^2(W_{I,XY}^{+})^2\}.\end{aligned} \tag{21}$$

The four boundary conditions at the stringer edge for the buckling state are obtained from the jump conditions by substituting the asymptotic expression (eqn (6)) into the jump conditions and then collecting the linear terms in ξ. These conditions can be simplified by employing the physical requirements that the displacements of the buckling mode U_1, V_1 and W_1 are either symmetric or anti-symmetric functions with respect to the stringer edge. Doing so yields

$$f_1(y=0)=0, \qquad f_{1,yy}(y=0)=0, \qquad w_1(y=0)=0. \tag{22–24}$$

The remaining jump condition with respect to $W_{1,Y}$ can be written as (using $W^+_{1,YY}-W^-_{1,YY}=2W^+_{1,YY}$)

$$(2D)W_{1,YY}(X,Y=0)+[G_sJ_s-(E_sA_s\bar{\sigma}e_s^2/E)]W_{1,XXY}(X,Y=0)=0. \tag{25}$$

Making use of separation of variables, this jump condition can be written in non-dimensional form

$$(M^2\gamma_s/2)w_{1,y}(y=0)-w_{1,yy}(y=0)=0. \tag{26}$$

The non-dimensional torsional-rigidity ratio γ_s is defined to be (Hui *et al.*, 1981; Hui, 1983)

$$\gamma_s=[q_0G_sJ_s/(DR)]-(E_n\alpha_s)(e_s/h)^2(8\pi c_1c^2\theta) \tag{27}$$

where the second term is usually small compared with the first term. Further, E_n is the ratio of Young's modulus of stringers to that of the skin, α_s is the area ratio, c_1 is a non-dimensional negative quantity being a function of the applied load (Hui, 1983) and θ is the flatness parameter defined by Koiter (1956) such that

$$E_n\alpha_s=(E_s/E)[A_s/(Bh)] \tag{28a}$$

$$c_1=\sigma(1+\alpha_s)[-h/(Rc)]/(1+E_n\alpha_s) \tag{28b}$$

$$\theta=q_0B/(2\pi R) \tag{29}$$

where E_s/E is the ratio of Young's modulus of the stringer to that of the skin, and B is the curved distance between adjacent stringers. Finally, the symmetry conditions at the mid-panel, defined to be the mid-point between adjacent stringers, are (Koiter, 1956; Stephens, 1971)

$$w_{1,y}(y=\pi\theta)=0, \quad w_{1,yyy}(y=\pi\theta)=0, \quad f_{1,y}(y=\pi\theta)=0, \quad f_{1,yyy}(y=\pi\theta)=0. \tag{30}$$

These two coupled ordinary differential equations for the buckling state are discretized using a central finite-difference scheme and the resulting eigenvalue problem is solved using the shifted inverse power method (Hui and Hansen, 1982).

3. SECOND-ORDER FIELDS

This section aims to derive the governing equilibrium and compatibility equations for the second-order fields and the associated boundary conditions. These partial differential equations are reduced to two sets of ordinary differential equations by using the separation of variables. The governing differential equations and the boundary conditions are discretized using a central finite-difference scheme and the resulting non-homogeneous algebraic equations are solved using backward substitution. The second-order fields are needed to compute the b coefficients (Budiansky and Hutchinson, 1964) of a structure which exhibits symmetric postbuckling behavior.

Substituting the asymptotic expansion (eqn (6)) into the governing non-linear differential equations and collecting the second-order terms involving ξ^2, the equilibrium and compatibility equations for the second-order fields are

$$w_{II,xxxx}+w_{II,yyyy}+2w_{II,xxyy}+f_{II,xx}+2\sigma w_{II,xx} = (2c)\,(f_{I,yy}w_{I,xx}+f_{I,xx}w_{I,yy}-2f_{I,xy}w_{I,xy}) \tag{31}$$

$$f_{II,xxxx}+f_{II,yyyy}+2f_{II,xxyy}-w_{II,xx} = (2c)\,[(w_{I,xy})^2-w_{I,xx}w_{I,yy}]. \tag{32}$$

Substituting the separable form of the buckling mode $w_I(x,y)$ and $f_I(x,y)$ into the right-hand side of the above partial differential equations, it is apparent that the second-order field may also be written in the separable form (Stephens, 1971; Hui *et al.*, 1981)

$$\begin{aligned} w_{II}(x,y) &= w^*(y)+w_A(y)\cos(2Mx) \\ f_{II}(x,y) &= f^*(y)+f_A(y)\cos(2Mx). \end{aligned} \tag{33}$$

The first set of ordinary differential equations for the (w^*, f^*) problem is

$$w^*(y)_{,yyyy} = -M^2[f_I(y)w_I(y)]_{,yy} \tag{34}$$

$$f^*(y)_{,yyyy} = (M^2/2)\,[w_I(y)^2]_{,yy}. \tag{35}$$

The second set of ordinary differential equations for the (w_A, f_A) problem is

$$\begin{aligned} w_A(y)_{,yyyy}-8M^2w_A(y)_{,yy}+(16M^4-8M^2\sigma)w_A(y)-4M^2f_A(y) \\ = (-M^2)\,[f_I(y)_{,yy}w_I(y)+f_I(y)w_I(y)_{,yy}-2f_I(y)_{,y}w_I(y)_{,y}] \end{aligned} \tag{36}$$

$$f_A(y)_{,yyyy}-8M^2f_A(y)_{,yy}+16M^4f_A(y)+4M^2w_A(y) = (-M^2)\,\{[w_I(y)_{,y}]^2-w_I(y)w_I(y)_{,yy}\}. \tag{37}$$

The boundary conditions at the edge of the stringer for the second-order fields are obtained by substituting the asymptotic expansion (eqn (6)) into the jump conditions and then collecting the terms which involve ξ^2. Since the second-order displacements are symmetric with respect to the edge of the stringer (Stephens, 1971), the two jump conditions with respect to $W_{,Y}$ and V are not applicable and should be replaced by, respectively

$$w_{II,y}(x, y=0) = 0 \tag{38}$$

$$v_{II}(x, y=0) = 0. \tag{39}$$

The first condition implies

$$w^*_{,y}(y=0) = 0, \qquad w_{A,y}(y=0) = 0. \tag{40, 41}$$

Solving for $V_{,XX}$ in the Donnell non-linear strain–displacement relations, and then making use of the asymptotic expansion and collecting terms which involve ξ^2, one obtains

$$(\varepsilon_x^{II})_{,Y}-2(\varepsilon_{xy}^{II})_{,X} = W_{I,X}W_{I,XY}-V_{II,XX}-W_{I,XX}W_{I,Y}-W_{I,X}W_{I,XY}. \tag{42}$$

Evaluating this expression at the edge of the stringer, the zero second-order circumferential displacement at the stringer edge condition can be written in terms of the stress function

$$f_{\text{II},yyy}(x, y = 0) + (2+\nu) f_{\text{II},xxy}(x, y = 0) = 0 \tag{43}$$

which implies

$$f_{\text{A},yyy}(y = 0) - 4M^2(2+\nu) f_{\text{A},y}(y = 0) = 0. \tag{44}$$

Employing the antisymmetry argument of the shear stress ($N^+_{xy_{\text{II}}} - N^-_{xy_{\text{II}}} = 2N^+_{xy_{\text{II}}}$), the jump condition with respect to U becomes

$$2N^+_{xy_{\text{II}}} + (E_s A_s)\{[1/(Eh)](F^+_{\text{II},YY} - \nu F^+_{\text{II},XX}) - e_s W^+_{\text{II},XX} + (1/2)(e_s)^2 (W^+_{\text{I},XY})^2\}_{,X} = 0 \tag{45}$$

or in non-dimensional form

$$\begin{aligned} -2f_{\text{II},xy}(y = 0) + (2\pi\theta)(E_n \alpha_s)\{f_{\text{II},yy}(y = 0) - \nu f_{\text{II},xx}(y = 0) \\ -2c(e_s/h) w_{\text{II},xx}(y = 0) + 2c^2(e_s/h)^2(h/R)[w_{\text{I},xy}(y = 0)]^2\}_{,x} = 0. \end{aligned} \tag{46}$$

Using the separable form of the second-order field, the boundary condition with respect to U for the $(w_{\text{A}}, f_{\text{A}})$ problem becomes

$$\begin{aligned} (4M) f_{\text{A},y}(y = 0) + (2\pi\theta)(E_n \alpha_s)\{-2M f_{\text{A},yy}(y = 0) - 8\nu M^3 f_{\text{A}}(y = 0) \\ -16cM^3(e_s/h) w_{\text{A}}(y = 0) + 2c^2 M^3 (e_s/h)^2 (h/R)[w_{\text{I},y}(y = 0)]^2\} = 0. \end{aligned} \tag{47}$$

Finally, upon using the condition $W^+_{\text{II},YYY} - W^-_{\text{II},YYY} = 2W^+_{\text{II},YYY}$ and the jump condition with respect to U that $N^{\text{II}}_{s,X} = 2F^+_{\text{II},XY}$, one obtains

$$(-2D) W^+_{\text{II},YYY} + (e_s)(2F^+_{\text{II},XY})_{,X} - E_s I_s W^+_{\text{II},XXXX} + A_s \bar{\sigma} W^+_{\text{II},XX} = 0 \tag{48a}$$

or in non-dimensional form

$$\begin{aligned} [q_0^3/(2c^2)][-w_{\text{II},yyy}(y = 0) + (2ce_s/h) f_{\text{II},xxy}(y = 0)] \\ -(2\pi\theta\beta_s/q_0)(R/h)^2 W_{\text{II},xxxx}(y = 0) + (4\pi\theta\sigma\alpha_s/q_0)(R/h)^2 w_{\text{II},xx}(y = 0) = 0 \end{aligned} \tag{48b}$$

where the out-of-plane bending stiffness ratio β_s is defined as $E_s I_s/(DB)$. Thus, one obtains

$$w^*_{,yyy}(y = 0) = 0 \tag{49}$$

$$\begin{aligned} [q_0^3/(2c^2)][-w_{\text{A},yyy}(y = 0) - 8M^2 c(e_s/h) f_{\text{A},y}(y = 0)] - [(32M^4\pi\theta\beta_s/q_0)(R/h)^2 \\ \times (16M^2\pi\theta\sigma\alpha_s/q_0)(R/h)^2] w_{\text{A}}(y = 0) = 0. \end{aligned} \tag{50}$$

The boundary conditions for the second-order fields at the mid-panel are similar to those for the buckling form

$$\begin{aligned} w_{\text{II},y}(x, y = \pi\theta) = 0, \qquad w_{\text{II},yyy}(x, y = \pi\theta) = 0 \\ f_{\text{II},y}(x, y = \pi\theta) = 0, \qquad f_{\text{II},yyy}(x, y = \pi\theta) = 0. \end{aligned} \tag{51}$$

It can be seen that the above boundary conditions are not sufficient to solve the (w^*, f^*) problem. Note that it is not necessary to solve for $f^*(y)$ explicitly since it will be shown that the postbuckling coefficient depends only on $f^*(y)_{,yy}$. Thus, an additional condition of zero average longitudinal second-order stress needs to be enforced in the form

$$\int_{Y=0}^{B}\int_{X=0}^{L} F_{\mathrm{II},YY}\,\mathrm{d}X\,\mathrm{d}Y+(E_sA_s)\int_{X=0}^{L}[\varepsilon_x^+-(e_s)W_{\mathrm{II},XX}+(1/2)(e_s)^2(w_{\mathrm{I},XY})^2]\,\mathrm{d}X=0 \tag{52}$$

or in non-dimensional form

$$\int_{y=0}^{2\pi\theta}\int_{x=0}^{x_0}\{f_{\mathrm{II},yy}+(E_n\alpha_s)(f^+_{\mathrm{II},yy}-\nu f^+_{\mathrm{II},xx})+(2cE_n\alpha_s)$$
$$\times[-(e_s/h)(w^+_{\mathrm{II},xx})+(e_s/h)^2(ch/R)(w^+_{\mathrm{I},xy})^2]\}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{53}$$

Substituting the separable form for $w_{\mathrm{II}}(x,y)$ and $f_{\mathrm{II}}(x,y)$ into the above equation and integrating over the length analytically, one obtains

$$\int_{y=0}^{2\pi\theta} f^*(y)_{,yy}\,\mathrm{d}y+(2\pi\theta)(E_n\alpha_s)f^*_{,yy}(y=0)+T=0 \tag{54}$$

where

$$T=(\pi\theta)(2c^2E_n\alpha_s)(e_s/h)^2(h/R)M^2[w_{\mathrm{I},y}(y=0)]^2. \tag{55}$$

Integrating eqn (35) twice with respect to y, one obtains

$$f^*(y)_{,yy}=(M^2/2)[w_{\mathrm{I}}(y)]^2+c_0 \tag{56}$$

which implies $f^*_{,yy}(y=0)=c_0$. Substituting $f^*(y)_{,yy}$ from eqn (56) into eqn (54), the constant of integration c_0 can be obtained from

$$(2\pi\theta)(1+E_n\alpha_s)(c_0)=(-M^2/2)\int_{y=0}^{2\pi\theta}[w_{\mathrm{I}}(y)]^2\,\mathrm{d}y-T. \tag{57}$$

The above sets of ordinary differential equations and the boundary conditions are sufficient to determine $w^*(y)$, $f^*(y)_{,yy}$, $w_{\mathrm{A}}(y)$ and $f_{\mathrm{A}}(y)$ which are necessary to compute the postbuckling coefficients.

4. KOITER'S IMPROVED POSTBUCKLING COEFFICIENTS

According to Koiter's postbuckling theory which was re-formulated by Budiansky and Hutchinson (1964), the present symmetric single-mode structure is sensitive to the presence of unavoidable geometric imperfection if the postbuckling coefficient b is negative, whereas the structure is not sensitive to imperfection for a positive b coefficient. The extent of imperfection-sensitivity depends on the magnitude of coefficient b. This section aims to derive the b coefficients for the integrally stiffened cylindrical panels as a function of the buckling mode $w_{\mathrm{I}}(x,y)$, $f_{\mathrm{I}}(x,y)$ and the second-order fields $w_{\mathrm{II}}(x,y)$, $f_{\mathrm{II}}(x,y)$ and the applied axial compressive load σ.

For a single-mode structure which exhibits symmetric postbuckling behavior, the equilibrium path of an imperfect system is specified by

$$b\xi^3+[1-(\sigma/\sigma_c)]\xi=(\sigma/\sigma_c)\bar{\xi}. \tag{58}$$

In the above, σ_c is the classical buckling load of the perfect system, ξ is the amplitude of the buckling mode normalized with respect to the out-of-plane buckling deflection at the mid-panel, and $\bar{\xi}$ is the amplitude of the geometric imperfection which is taken to be of the same shape as the buckling mode. The imperfection-sensitivity curve is governed by (σ^* is the buckling load of an imperfect system)

$$|\bar{\xi}| = \frac{2[1-(\sigma^*/\sigma_c)]^{3/2}}{3(-3b)^{1/2}(\sigma^*/\sigma_c)} \tag{59}$$

which is valid only if b is negative. Equations (58) and (59) remain valid within Koiter's improved postbuckling theory except that coefficient b is a function of the applied load.

The postbuckling coefficient b for an integrally stiffened cylindrical panel under compression is defined to be (Hutchinson and Amazigo, 1967; Tvergaard, 1973)

$$b = (2Q_1 + Q_{2a} + Q_{2b})/|Q_0| \tag{60}$$

where (L is the length of the shell)

$$Q_1 = \int_{Y=0}^{B}\int_{X=0}^{L} [F_{1,YY}(W_{1,X}W_{11,X}) + F_{1,XX}(W_{1,Y}W_{11,Y}) - F_{1,XY}(W_{1,X}W_{11,Y} + W_{1,Y}W_{11,X})]\,\mathrm{d}X\,\mathrm{d}Y + \int_{X=0}^{L} N_s^{1} L_{11}^{r}(U_1, U_{11})\,\mathrm{d}X \tag{61}$$

$$Q_{2a} = \int_{Y=0}^{B}\int_{X=0}^{L} [F_{11,YY}(W_{1,X})^2 + F_{11,XX}(W_{1,Y})^2 - 2F_{11,XY}(W_{1,X}W_{1,Y})]\,\mathrm{d}X\,\mathrm{d}Y \tag{62}$$

$$Q_{2b} = \int_{X=0}^{L} N_s^{11} L_2^{r}(U_1)\,\mathrm{d}X \tag{63}$$

$$Q_0 = \int_{Y=0}^{B}\int_{X=0}^{L} (h\bar{\sigma})(W_{1,X})^2\,\mathrm{d}X\,\mathrm{d}Y + \int_{X=0}^{L} |N_s^{p}|\,L_2^{r}(U_1)\,\mathrm{d}X. \tag{64}$$

In the above expressions, N_s^p, N_s^1 and N_s^{11} are defined in eqns (21) and the quantities $L_2^r(U_1)$ and $L_{11}^r(U_1, U_{11})$ are obtained from eqn (18b) such that (the superscript "r" refers to the reinforced-stringer)

$$\varepsilon_s = L_1^r(U) + (1/2)L_2^r(U) \tag{65}$$

where

$$L_1^r(U) = U_{,X}^{+} - e_s W_{,XX}^{+} \tag{66}$$

$$L_2^r(U) = (W_{,X}^{+})^2 + (e_s)^2(W_{,XY}^{+})^2. \tag{67}$$

Using Budiansky–Hutchinson's notation, one obtains, upon substituting the asymptotic expansion $W = \xi W_1 + \xi^2 W_{11}$

$$L_2^r(U_1 + U_{11}) = L_2^r(U_1) + L_2^r(U_{11}) + 2L_{11}^r(U_1, U_{11}). \tag{68}$$

Assuming that the stringer does not bend at the buckling state (so that $W_{1,X}^{+} = 0$) and the stringer does not twist at the second-order state ($W_{11,XY}^{+} = 0$) one obtains

$$L_2^r(U_1) = (e_s)^2(W_{1,XY}^{+})^2, \qquad L_{11}^r(U_1, U_{11}) = 0. \tag{69, 70}$$

Making use of the asymptotic expansion (eqn (6)) and integrating the expressions in the axial direction analytically, Q_1, Q_{2a}, Q_{2b} and Q_0 can be written in non-dimensional form with the common factor $Eh^4x_0M^2/R$ (note that $x_0 = q_0L/R$) such that

$$Q_1 = (Eh^4x_0M^2/R)\int_{y=0}^{2\pi\theta}\{(1/2)w_1(y)f_1(y)_{,yy}w_A(y) - f_1(y)w_1(y)_{,y}[(1/2)w^*(y)_{,y} + (1/4)w_A(y)_{,y}] - f_1(y)_{,y}w_1(y) \times [(1/2)w^*(y)_{,y} - (1/4)w_A(y)_{,y}] - (1/2)f_1(y)_{,y}w_1(y)_{,y}w_A(y)\}\,dy \quad (71)$$

$$Q_{2a} = (Eh^4x_0M^2/R)\int_{y=0}^{2\pi\theta}\{w_1(y)^2[(1/2)f^*(y)_{,yy} - (1/4)f_A(y)_{,yy}] - [w_1(y)_{,y}]^2 f_A(y) - w_1(y)w_1(y)_{,y}f_A(y)_{,y}\}\,dy \quad (72)$$

$$Q_{2b} = (Eh^4x_0M^2/R)(\pi\theta E_n\alpha_s)(4ch/R)(e_s/h)^2[w_{1,y}(y=0)]^2 \times \{(1/2)f^*_{,yy}(y=0) - (1/4)f_{A,yy}(y=0) - M^2\nu f_A(y=0) - (2cM^2e_s/h)w_A(y=0) + (3M^2/4)(c^2h/R)(e_s/h)^2[w_{1,y}(y=0)]^2\} \quad (73)$$

$$Q_0 = (Eh^4x_0M^2/R)(\sigma/c)\left\{(1/2)\int_{y=0}^{2\pi\theta} w_1(y)^2\,dy + (2cE_n\alpha_s\pi\theta h/R)[w_{1,y}(y=0)]^2\right\}. \quad (74)$$

The circumferential integrations are performed using Simpson's rule with approximately 101 integration points between adjacent stringers. It should be noted that Q_{2b} and the second term in the Q_0 are usually small compared with the rest of the terms in the expression for coefficient b.

5. DISCUSSION OF RESULTS

The flatness parameter θ is the most important parameter which characterizes the geometry of the structure. It combines the radius-to-thickness ratio and the width-to-radius ratio into one parameter. The present imperfection-sensitivity problem depends also on the cross-sectional area ratio $E_n\alpha_s$, the torsional-rigidity ratio γ_s and to a smaller extent on the eccentricity ratio e_s/h. The length of the cylindrical panel is infinite and Poisson's ratio is assumed to be 0.3.

Figure 1(a) shows a graph of the classical buckling load vs the flatness parameter for long cylindrical panels under axial compression. For a given value of θ, the torsional-rigidity of the stringer may considerably raise the classical buckling load. The $\gamma_s = 0$ curve corresponds to simply-supported cylindrical panels while $\gamma_s \geqslant 100$ corresponds to clamped edges. Note that the classical buckling load is approximately independent of the area ratio and the eccentricity ratio and these two parameters are set to zero in this figure.

Figure 1(b) shows the corresponding postbuckling coefficient b where $E_n\alpha_s = 0$ and $e_s/h = 0$. For simply-supported cylindrical panels, the coefficient b is positive for $0 \leqslant \theta < 0.648$ while it is negative for $0.648 < \theta \leqslant 1.0$ and the transitional value 0.648 agrees with that obtained by Koiter (1956). The "improved" b coefficient is also plotted and it appears that the imperfection sensitivity of the cylindrical panels predicted by the more accurate "improved" theory may be far less serious than that predicted using Koiter's 1945 general theory. The general curve and the improved curves practically coincide if the b coefficient turns out to be positive. Similar curves for the clamped cylindrical panels are also plotted. Substantial positive shifts of the postbuckling coefficients b are found using Koiter's improved theory and this conclusion appears to be in qualitative agreement with experiments (Tennyson *et al.*, 1980). Typical imperfection-sensitivity curves are presented in Fig. 2 using Koiter's 1945 general theory and Koiter's improved theory. The results based on a least square curve-fit are also presented.

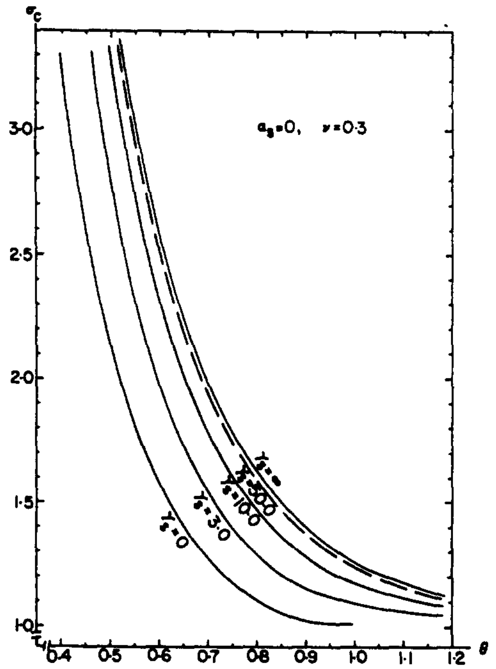

Fig. 1(a). Classical buckling load vs the flatness parameter for integrally stiffened cylindrical panels under compression.

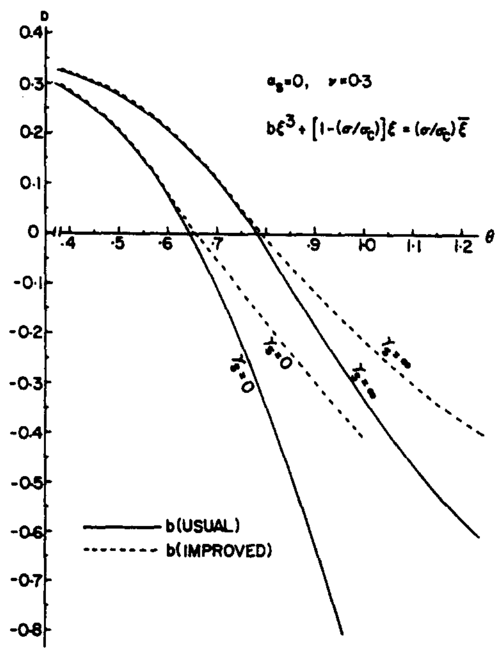

Fig. 1(b). The corresponding postbuckling coefficients b vs the flatness parameter.

As an example problem, equally-spaced integrally-stiffened closed cylindrical shells are chosen where the local buckling and postbuckling behavior of the panel between adjacent stringers is of interest. The cross-sectional shape of each stringer is rectangular and the following parameters are held fixed (Byskov and Hutchinson, 1977; Hui *et al.*, 1981; Hui, 1983)

$$E_n = 1.0, \quad \alpha_s = 0.7, \quad R/h = 850, \quad \nu = 0.3, \quad q_0 = 52.998. \tag{75}$$

As the number of stringers increases (decreases), the stringer eccentricity decreases (increases) such that the area $E_n \alpha_s$ and the tangential width of the stringer remains constant.

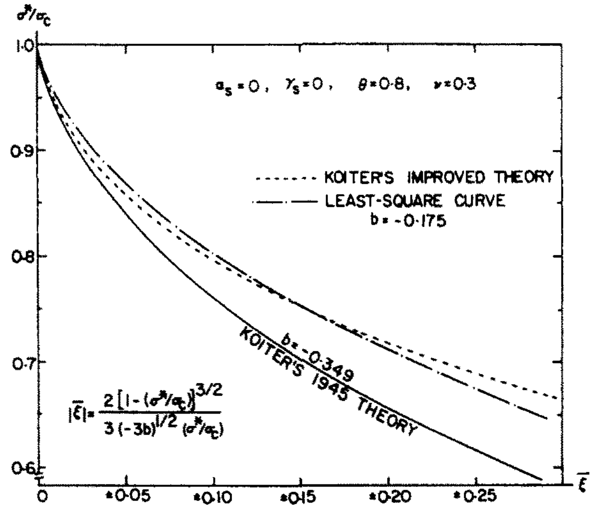

Fig. 2. Typical inperfection-sensitivity curves using Koiter's 1945 general theory, Koiter's improved postbuckling theory and the curve-fit results.

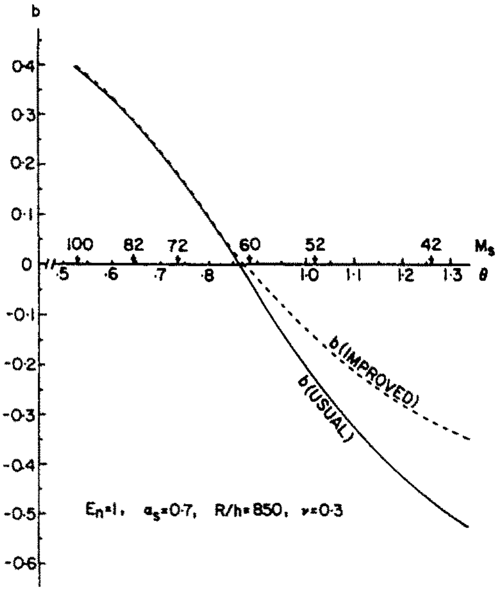

Fig. 3. Postbuckling coefficients b vs the flatness parameter for the example problem.

Doing so, the flatness parameter θ, the eccentricity ratio e_s/h, and the torsional rigidity ratio γ_s are permitted to vary. The postbuckling coefficients b predicted using Koiter's 1945 general theory and Koiter's improved theory are plotted in Fig. 3. The transitional value of coefficient b appears to be approximately 0.86. It may be appreciated that the reduction of the imperfection-sensitivity (as measured by the positive shift of coefficient b) may be quite significant. Detailed numerical results are presented in Table 1.

6. CONCLUSIONS

The imperfection-sensitivity behavior of integrally stringer-reinforced cylindrical panels under compression has been investigated. It is found that the positive shift of the postbuckling coefficient may be quite significant due to the evaluation of these coefficients at the actual applied load rather than at the classical buckling load. The positive shift is especially apparent for larger values of the flatness parameter. It is strongly recommended that Koiter's "improved" postbuckling theory be used unless coefficient b turns out to be positive or the computational effort involved becomes prohibitive. The present results show that virtually "all" the imperfection-sensitivity curves predicted using Koiter's general

Table 1. Parameter variations of equally-spaced integrally stiffened cylindrical panels under axial compression

No. of stringers	θ	$2M\theta$	γ_s	σ_c	b	b (improved)
100	0.52999	1.472	37.58	3.1264	0.39183	0.39810
94	0.5638	1.4787	40.79	2.7936	0.36739	0.37264
90	0.5889	1.4845	43.51	2.5858	0.34698	0.35142
86	0.6163	1.4911	46.19	2.3885	0.32196	0.32550
82	0.6463	1.4995	49.2	2.2030	0.29160	0.29420
77	0.6883	1.5131	53.36	1.9872	0.24400	0.24539
72	0.7361	1.5323	58.16	1.7913	0.18323	0.18352
68	0.7794	1.5533	62.74	1.6491	0.12282	0.12254
64	0.8281	1.5821	67.34	1.5209	0.05052	0.05016
60	0.8833	1.6231	72.81	1.4075	−0.03402	−0.01092
55	0.9636	1.7009	80.52	1.2877	−0.15429	−0.09143
52	1.0192	1.7690	85.75	1.2281	−0.23094	−0.14399
50	1.0600	1.8265	89.66	1.1935	−0.28215	−0.17925
47	1.1276	1.9345	95.72	1.1489	−0.35734	−0.23174
45	1.1777	2.0233	100.58	1.1240	−0.40544	−0.26552
42	1.2619	2.1842	108.14	1.0928	−0.47405	−0.31401
40	1.325	2.3115	113.86	1.0759	−0.51758	−0.34455

theory reported in the open literature tend to overestimate the degrading effects of imperfections. The extent of such an overestimation will, of course, depend on the particular instability problem under consideration. Extension of the present work to laminated cylindrical shells (such as those examined by Zhang and Matthews (1983) and Hui (1985a) is in progress. Application of Koiter's improved theory to buckling of beams on elastic foundations (Hui, 1986, 1987) will be published in separate papers.

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